Determining the Neutrinos Mass Hierarchy from Cosmological Data

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Neutrinos Mass Hierarchy

In 1930, to save energy conservation in β-decays, Wolfgang Pauli confessed to the astronomer Walter Baade that

I have done a terrible thing today, something which no theoretical physicist should ever do. I have suggested something that can never be verified experimentally.

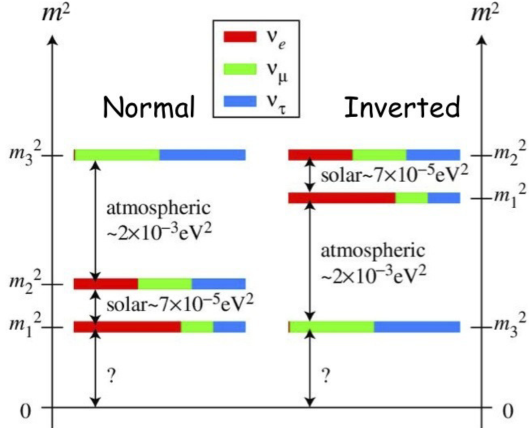
However, just after less than 30 years, Reines and Cowan discovered the neutrinos. And now the accelerator experiments have also confirmed that there are **only three** generations of light neutrinos which are weakly interacting:

Generation	1st	2nd	3rd
quarks	u	c	t
	d	s	b
leptons	ν_e	ν_μ	ν_τ
	e^-	μ^-	τ^-

Fermion IRs under $SU(2)_L \times U(1)_Y$			I	I_3	Y	Q
$L_{eL} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	$L_{\mu L} \equiv \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$L_{\tau L} \equiv \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$	1/2	$1/2$ $-1/2$	-1	0 -1
$l_{eR} \equiv e_R$	$l_{\mu R} \equiv \mu_R$	$l_{\tau R} \equiv \tau_R$	0	0	-2	-1
$Q_{1L} \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$Q_{2L} \equiv \begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$Q_{3L} \equiv \begin{pmatrix} t_L \\ b_L \end{pmatrix}$	1/2	$1/2$ $-1/2$	1/3	$2/3$ $-1/3$
$q^U_{uR} \equiv u_R$	$q^U_{cR} \equiv c_R$	$q^U_{tR} \equiv t_R$	0	0	4/3	2/3
$q^D_{dR} \equiv d_R$	$q^D_{sR} \equiv s_R$	$q^D_{bR} \equiv b_R$			$-2/3$	$-1/3$

But, up to now, we still do not understand the neutrinos **mass spectrum** and the real nature of neutrinos as **Dirac or Majorana particles.**

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = 7.5\times10^{-5}\ \mathrm{eV}^2, \quad |\Delta m_{31}^2| \equiv |m_3^2 - m_1^2| = 2.5\times10^{-3}\ \mathrm{eV}^2$$



K.A. Olive et al. (Particle Data Group), Chin. Phys. C 38, 090001 (2014).

$\sum m_\nu < 0.12$ eV 95%,Planck TT,TE,EE+lowE +lensing+BAO

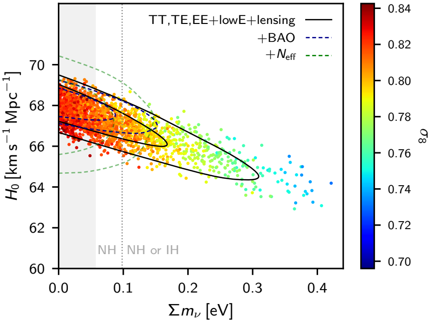


Figure: Planck 2018: The grey band on the left shows the region with $\sum m_\nu < 0.056\,$eV ruled out by neutrino oscillation experiments. Mass splittings observed in neutrino oscillation experiments also imply that the region left of the dotted vertical line can only be a normal hierarchy (NH), while the region to the right could be either the normal hierarchy or an inverted hierarchy (IH).

Credits: Planck 2018 results. VI. Cosmological parameters, arXiv:1807.06209 [astro-ph.CO]

How could/did we study neutrinos in cosmology?

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Neutrinos play an significant role in cosmology:
(1) Neutrinos interact with charged leptons and hadrons in the primordial plasma via electroweak interactions.

$$\begin{aligned} \nu_e + n &\to e^- + p & \qquad \bar{\nu}_e + p &\to e^+ + n \\ e^- + p &\to \nu_e + n & \qquad n &\to e^- + \bar{\nu}_e + p \\ e^+ + n &\to \bar{\nu}_e + p & \qquad e^- + \bar{\nu}_e + p &\to n, \end{aligned}$$

(2) Neutrinos couple to gravity and contribute to the energy density via Einstein equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, at the background and the first order perturbation levels.
(3) Neutrinos decouple and propagate freely along geodesic lines, once the interaction rate becomes lower than the expansion rate.

Therefore, on the contrary, cosmology also plays an important role to reveal the nature of neutrinos via the imprints on the sky, for example CMB power spectrum and matter power spectrum.

Big Bang Nucleosynthesis

BBN Relation

In the early universe with a higher temperature than O(1) MeV the inter-converting reactions between neutron and proton through the weak interaction ($n + e^+ \leftrightarrow p + \nu_e$, $n + \bar{\nu}_e \leftrightarrow p + e^-$, and $n \leftrightarrow p + e^- + \nu_e$) were sufficiently rapid. In this case the neutron to proton ratio obeys its thermal equilibrium value,

$$\frac{n}{p} = \exp\left[-\frac{\Delta m_{np} + \mu_{\nu_e}}{T}\right] = \exp\left[-\frac{\Delta m_{np}}{T} - \xi_{\nu_e}\right], \quad (1)$$

with the mass difference $\Delta m_{np} = 1.3$ MeV. Here we explicitly wrote the degeneracy parameter of ν_e to be $\xi_{\nu_e} = \mu_{\nu_e}/T_\nu$ with μ_{ν_e} being the chemical potential of ν_e. It is remarkable that the electron's chemical potential μ_{e^-} must be much smaller than that of ν_e because of the neutrality of the Universe $\xi_e = \mu_{e^-}/T \sim O(\eta) \ll \xi_{\nu_e}$ with T and η being the photon temperature and the baryon-to-photon ratio, respectively. Accordingly ξ_{ν_e} affects the freezeout value of n/p, which can change the light element abundances. In particular ^{4}He mass fraction Y_p depends on ξ_{ν_e} in addition to η (or $\Omega_b h^2$) and N_ν. Thus Y_p is related to those three parameters i.e., Y_p=$Y_p(\Omega_b h^2,\ \xi_{\nu_e},\ \Delta N_\nu)$, which is called "the BBN relation." But, in this work we take $N_{eff} \equiv 3 + \frac{30}{7}\left(\frac{\xi_\nu}{\pi}\right)^2 + \frac{15}{7}\left(\frac{\xi_\nu}{\pi}\right)^4 = 3.046$ and $\Delta N_\nu = 0$.

Neutrinos Effects on CMB Power Spectrum

Neutrinos at Background

The distribution function for ν_i and $\bar{\nu}_i$ $(i = e, \mu, \tau)$ are given by

$$f_{\nu_i}(p_i) = \frac{1}{e^{p_i/T_\nu + \xi_{\nu i}} + 1}, \qquad f_{\bar{\nu}_i}(p_i) = \frac{1}{e^{p_i/T_\nu - \xi_{\nu i}} + 1}, \tag{2}$$

where p_i is momentum of ν_i. $\xi_{\nu i} \equiv \mu_{\nu i}/T_\nu$ is the degeneracy parameter, $\mu_{\nu i}$ being the chemical potential. T_ν is the temperature of neutrino and related to that at the present epoch T_{ν_0} as $T_\nu = T_{\nu 0}/a$ with a being the scale factor. The energy density and pressure of a neutrino species are given by

$$\rho_\nu + \rho_{\bar{\nu}} = \frac{1}{2\pi^2} \int_0^\infty p^2 dp \sqrt{p^2 + m^2} \left(f_\nu + f_{\bar{\nu}}\right), \tag{3}$$

$$p_\nu + p_{\bar{\nu}} = \frac{1}{2\pi^2} \int_0^\infty p^2 dp \frac{p^2}{3\sqrt{p^2 + m^2}} \left(f_\nu + f_{\bar{\nu}}\right). \tag{4}$$

where the subscript i is omited for simplicity.

Using the comoving momentum $q \equiv pa$, one has the above integral

$$\rho_\nu + \rho_{\bar{\nu}} = \frac{T_\nu^4}{2\pi^2}\int_0^\infty y^3 dy \sqrt{1+\left(\frac{a\tilde{m}}{y}\right)^2}\left(\frac{1}{e^{y+\xi}+1}+\frac{1}{e^{y-\xi}+1}\right), \quad (5)$$

$$p_\nu + p_{\bar{\nu}} = \frac{T_\nu^4}{6\pi^2}\int_0^\infty y^3 dy \frac{1}{\sqrt{1+(a\tilde{m}/y)^2}}\left(\frac{1}{e^{y+\xi}+1}+\frac{1}{e^{y-\xi}+1}\right) \quad (6)$$

where y and $\tilde{m}$ are defined as

$$y \equiv \frac{q}{T_{\nu 0}}, \qquad \tilde{m} \equiv \frac{m}{T_{\nu 0}}. \quad (7)$$

In principle, the above integrals can be calculated numerically, but if $|\xi| \ll \mathcal{O}(1)$ is respected, one can obtain useful approximation in relativistic and non-relativistic limits.

Relativistic limit

When $\frac{a\tilde{m}}{y}(=\frac{m}{p}) \ll 1$, up to the 2nd order in $\frac{a\tilde{m}}{y}$, the energy density and pressure can be written as

$$\rho_\nu + \rho_{\bar{\nu}} \simeq \frac{T_\nu^4}{2\pi^2}\int_0^\infty y^3 dy \left(1+\frac{1}{2}\left(\frac{a\tilde{m}}{y}\right)^2\right)\left(\frac{1}{e^{y+\xi}+1}+\frac{1}{e^{y-\xi}+1}\right), \quad (8)$$

$$p_\nu + p_{\bar{\nu}} \simeq \frac{T_\nu^4}{6\pi^2}\int_0^\infty y^3 dy \left(1-\frac{1}{2}\left(\frac{a\tilde{m}}{y}\right)^2\right)\left(\frac{1}{e^{y+\xi}+1}+\frac{1}{e^{y-\xi}+1}\right). \quad (9)$$

These integrals can be performed exactly and we obtain

$$\rho_\nu + \rho_{\bar{\nu}} \simeq \frac{7\pi^2}{120}T_\nu^4\left[\left\{1+\frac{30}{7}\left(\frac{\xi}{\pi}\right)^2+\frac{15}{7}\left(\frac{\xi}{\pi}\right)^4\right\}+\frac{5}{7\pi^2}(a\tilde{m})^2\left\{1+3\left(\frac{\xi}{\pi}\right)^2\right\}\right], \quad (10)$$

$$p_\nu + p_{\bar{\nu}} \simeq \frac{1}{3}\frac{7\pi^2}{120}T_\nu^4\left[\left\{1+\frac{30}{7}\left(\frac{\xi}{\pi}\right)^2+\frac{15}{7}\left(\frac{\xi}{\pi}\right)^4\right\}-\frac{5}{7\pi^2}(a\tilde{m})^2\left\{1+3\left(\frac{\xi}{\pi}\right)^2\right\}\right]. \quad (11)$$

Non-relativistic limit

When $\frac{a\tilde{m}}{y}(=\frac{m}{p}) \gg 1$, By taking into account the terms up to the 10th order in ξ, we obtain

$$\begin{aligned}\rho_\nu + \rho_{\bar{\nu}} \quad \simeq \quad & \frac{T_\nu^4}{2\pi^2}(a\tilde{m})\left[3\zeta(3) + (\log 4)\xi^2 + \frac{1}{24}\xi^4 - \frac{1}{1440}\xi^6 + \frac{1}{40320}\xi^8 - \frac{17}{14515200}\xi^{10}\right] \\ & + \frac{T_\nu^4}{4\pi^2}\frac{1}{a\tilde{m}}\left[45\zeta(5) + 18\zeta(3)\xi^2 + (\log 4)\xi^4 + \frac{1}{60}\xi^6 - \frac{1}{6720}\xi^8 + \frac{1}{302400}\xi^{10}\right],\end{aligned} \tag{12}$$

where $\zeta(x)$ means the Riemann zeta function. Similar calculations also hold for the pressure, and we have, up to the 10th order in ξ,

$$\begin{aligned}p_\nu + p_{\bar{\nu}} \quad \simeq \quad & \frac{T_\nu^4}{6\pi^2}\frac{1}{a\tilde{m}}\left[45\zeta(5) + 18\zeta(3)\xi^2 + (\log 4)\xi^4 + \frac{1}{60}\xi^6 - \frac{1}{6720}\xi^8 + \frac{1}{302400}\xi^{10}\right] \\ - \quad & \frac{T_\nu^4}{12\pi^2}\left(\frac{1}{a\tilde{m}}\right)^3\left[\frac{2835\zeta(7)}{2} + 675\zeta(5)\xi^2 + 45\zeta(3)\xi^4 + (\log 4)\xi^6 + \frac{1}{112}\xi^8 - \frac{1}{20160}\xi^{10}\right].\end{aligned} \tag{13}$$

Neutrinos at Perturbation

CMB physics mainly depends on the behavior of three species until photon decoupling: baryons, electrons and photons. Other species, such as neutrinos, affect CMB anisotropies in two ways: cosmological background evolution and contribution to the local gravitational forces. Thus, one should study the perturbation equations. In the Newtonian gauge in a spatially flat Universe, the perturbed line element is

$$ds^2 = a^2(\eta)\left[-(1+2\Psi(\eta,\vec{x}))d\eta^2 + (1-2\Phi(\eta,\vec{x}))d\vec{x}^2\right]. \tag{14}$$

The photon phase-space distribution function is written as

$$f_\gamma(\eta,\vec{x},\vec{p}) = \left[\exp\left(\frac{y}{a(\eta)\overline{T}(\eta)\left\{1+\Theta_\gamma(\eta,\vec{x},\hat{n})\right\}}\right)-1\right]^{-1}, \tag{15}$$

where $\Theta_\gamma(\eta,\vec{x},\hat{n}) \equiv \delta T/\overline{T}$ is the relative photon temperature shift. After some algebra, one has the Boltzmann equation

$$\Theta_\gamma^{'} + \hat{n}\cdot\vec{\nabla}\Theta_\gamma - \Phi^{'} + \hat{n}\cdot\vec{\nabla}\Psi = an_e\sigma_T(\Theta_{\gamma 0} - \Theta_\gamma + \hat{n}\cdot\vec{v}_B). \tag{16}$$

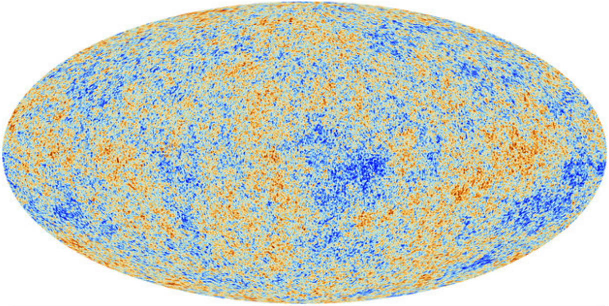
For the tightly coupling photons, performing the Legendre transformation of $\Theta_\gamma(\eta, \vec{x}, \hat{n})$ with respect to $\mu \equiv \hat{n} \cdot \hat{k}$ in Fourier space

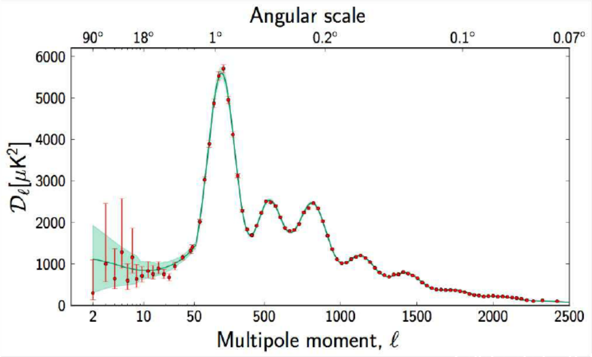
$$\Theta_\gamma(\eta, \vec{k}, \mu) = \sum_l (-i)^l (2l+1) \Theta_\gamma(\eta, \vec{k}) P_l(\mu). \tag{17}$$

Now, the Boltzmann equation leads to an infinite hierarchy of coupled equations

$$\begin{aligned}
\Theta'_{\gamma 0} &= -k\Theta_{\gamma 1} + \Phi', \\
\Theta'_{\gamma 1} &= \frac{k}{3}(\Theta_{\gamma 0} - 2\Theta_{\gamma 2} + \Psi) + a n_e \sigma_T \left(\frac{\theta_B}{3k} - \Theta_{\gamma 1} \right), \\
\Theta'_{\gamma l} &= \frac{k}{2l+1}[l\Theta_{\gamma(l-1)} - (l+1)\Theta_{\gamma(l+1)}] - a n_e \sigma_T \Theta_{\gamma l}, l \geq 2. && (18) \\
\frac{\delta T}{\overline{T}} &= \sum_{l,m} a_{lm} Y_{lm}(\hat{n}), \qquad a_{lm}(-i)^l \int \frac{d^3k}{2\pi^2} Y_{lm}(\hat{k}) \Theta_{\gamma l}(\eta_0, \vec{k}), && (19) \\
C_l &= \langle a_{lm} a^*_{lm} \rangle = \frac{1}{2\pi^2} \int \frac{dk}{k} [\Theta_{\gamma l}(\eta_0, \vec{k})]^2 \mathcal{P}_{\mathcal{R}}(k), && (20) \\
C_l^{obs} &= \frac{1}{2l+1} \sum_{-l \leq m \leq l} |a^{obs}_{lm}|^2. && (21)
\end{aligned}$$

where $\mathcal{P}_{\mathcal{R}}(k) = A_s(k/k_0)^{n_s - 1}$ is the primordial curvature spectrum.




Angular scale
90°
18°
1°
0.2°
0.1°
0.07°
6000
5000
4000
3000
2000
1000
0
$\mathcal{D}_\ell[\mu K^2]$
2
10
50
500
1000
1500
2000
2500
Multipole moment, ℓ

For neutrinos, the relative fluctuation of the phase space is defined as

$$\Psi_\nu(\eta, \vec{x}, q, \hat{n}) = \frac{f_\nu(\eta, \vec{x}, q, \hat{n})}{f_{\nu 0}(\eta, q)} - 1, \tag{22}$$

in the case of standard neutrinos with Fermi-Dirac distribution $f_\nu(\eta, \vec{x}, q, \hat{n}) = f_{\nu 0}(q/[a\overline{T}_\nu \{1 + \Theta_\nu(\eta, \vec{x}, \hat{n})\}])$, one has the first oder expansion in Θ_ν

$$\Psi_\nu(\eta, \vec{x}, q, \hat{n}) = -\Theta_\nu(\eta, \vec{x}, \hat{n}) \frac{d \ln f_{\nu 0}(q)}{dq}, \tag{23}$$

In replacing $f_\nu(\eta, \vec{x}, q, \hat{n})$ as a function of $\Psi_\nu(\eta, \vec{x}, q, \hat{n})$ in Boltzmann equation,

$$\Psi_\nu' + \frac{q}{\epsilon} \hat{n} \cdot \vec{\nabla} \Psi_\nu + \frac{d \ln f_{\nu 0}}{d \ln q} \left[\Phi' - \frac{\epsilon}{q} \hat{n} \cdot \vec{\nabla} \Psi \right] = 0, \tag{24}$$

where $\epsilon \equiv aE = \sqrt{q^2 + a^2 m^2}$.

Similar as photons, expanding Ψ_ν in Legendre polynomials in Fourier space, one obtains an infinite hierarchy of equations

$$\begin{aligned}\Psi'_{\nu 0} &= -\frac{qk}{\epsilon}\Psi_{\nu 1} - \Phi'\frac{d\ln f_{\nu 0}}{dq},\\ \Psi'_{\nu 1} &= \frac{qk}{3\epsilon}(\Psi_{\nu 0} - 2\Psi_{\nu 2}) - \frac{qk}{3\epsilon}\Psi\frac{d\ln f_{\nu 0}}{dq},\\ \Psi'_{\nu l} &= \frac{qk}{(2l+1)\epsilon}\left[l\Psi_{\nu(l-1)} - (l+1)\Psi_{\nu(l+1)}\right], l \geq 2. \end{aligned} \quad (25)$$

Integrating $\Psi_{\nu l=0,1,2}$, one has the perturbed stress-energy tensor

$$\begin{aligned}\delta\rho_\nu = \overline{\rho}_\nu\delta_\nu &= 4\pi a^{-4}\int q^2 dq\epsilon f_{\nu 0}(q)\Psi_0,\\ \delta P_\nu &= \frac{4\pi}{3}a^{-4}\int q^2 dq\frac{q^2}{\epsilon}f_{\nu 0}(q)\Psi_0,\\ (\overline{\rho}_\nu + \overline{P}_\nu)\theta_\nu &= 4\pi a^{-4}\int q^2 dq\epsilon f_{\nu 0}(q)\Psi_1,\\ (\overline{\rho}_\nu + \overline{P}_\nu)\sigma_\nu &= 4\pi a^{-4}\int q^2 dq\frac{q^2}{\epsilon}f_{\nu 0}(q)\Psi_2. \end{aligned} \quad (26)$$

contribute to the Poisson equation $-k^2\Phi = -4\pi a^2 G\sum_i \delta\rho_i$ and the potential slips $k^2(\Phi - \Psi) = 12\pi G a^2\sum_i(\overline{\rho}_i + \overline{p}_i)\sigma_i$

Neutrinos Effects on CMB Power Spectrum

Because gravity is sensitive to the total neutrino mass $M_\nu = \Sigma m_\nu$.

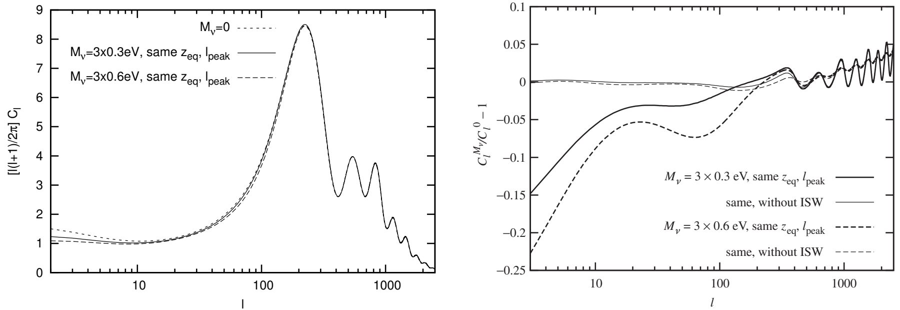


Figure: CMB temperature spectrum for models with $M_\nu = 3 \times 0.3$eV or $M_\nu = 3 \times 0.6$eV. The differences can be caused by $z_{eq} = \frac{\omega_M}{\omega_\gamma[1+\frac{7}{8}(\frac{4}{11})^{4/3}N_{eff}]}$, $d_s(\eta_{LS})/d_A(\eta_{LS})$, $d_d(\eta_{LS})/d_A(\eta_{LS})$ and damping $\exp[-(l/l_d)^2]$ with $l_d \sim k_d(\eta_0 - \eta_{LS})$ and $k_d \equiv 2\pi/r_d$. Here the difference is mainly due to the LISW effect and EISW effect, gravitationally driven oscillations effect and diffusion damping.*J. Lesgourgues, et al., Cambridge, UK: Cambridge University Press, 2013.*

Neutrinos Effect on the Large Scale Structures

For a fluid i with constant sound speed dominating the expansion of the Universe, the continuity, Euler, Friedmann and Poisson equations, the density contrast evolves on sub-Hubble scales as

$$\delta_i^{''} + \mathcal{H}\delta_i^{'} + \left(k^2 - \frac{3\mathcal{H}^2}{2c_s^2}\right) c_s^2 \delta_i = 0, \qquad (27)$$

where the effective mass $(k^2 - k_J^2)c_s^2$ changes sign when k equals to the Jeans wavenumber. Therefore for neutrinos, the free-streaming length is defined by analogy with the Jeans length

$$\lambda_{fs}(\eta) = a(\eta)\frac{2\pi}{k_{fs}} = 2\pi\sqrt{\frac{2}{3}}\frac{c_\nu(\eta)}{H(\eta)}. \qquad (28)$$

Below free-streaming length collisionless particles cannot remain confined in gravitational potential wells due to the velocity dispersion. For the nonrelativistic transition, the velocity dispersion is

$$c_\nu = \frac{\langle p \rangle}{m_\nu} = 3.15\frac{T_\nu}{m_\nu} = 158(1+z)(\frac{1\text{eV}}{m_\nu})\text{km s}^{-1} \qquad (29)$$

After the non-relativistic transition, the free streaming scale of neutrino changes from the Hubble scale to

$$k_{fs} = 0.776(1+z)^{-2}\frac{H(z)}{H_0}\left(\frac{m_\nu}{1\text{eV}}\right) h\text{Mpc}^{-1}, \tag{30}$$

which has a minimum at $z = z_{nr} = (\frac{m_\nu}{5.28\times 10^{-4}\text{eV}}) - 1$

$$k_{nr} = k_{fs}(z_{nr}) \approx 0.0178\Omega_m^{1/2}\left(\frac{m_\nu}{1\text{eV}}\right)^{1/2} h\text{Mpc}^{-1}, \tag{31}$$

in the matter domination, considering $H(z)/H_0 \approx \Omega_m^{1/2}(1+z)^{3/2}$. Here, the scale k_{nr} is the largest scale that can be affected by the presence of neutrino fluctuation. On smaller scales $k > k_{nr}$, density fluctuations are washed out, while on larger scales $k < k_{nr}$ neutrino behaves as cold dark matter.

Thus, at sufficiently small scales $k \gg k_{nr}$, the power spectrum of matter $P(k)$ is depressed due to the lack of neutrino power. This is due to the modification of the linear evolution of density perturbation at small scales $k \gg k_{nr}$ by the presence of the neutrino

$$\delta'' + 2\mathcal{H}\delta' = 4\pi G\rho a^2(1 - f_\nu)\delta, \tag{32}$$

where $\delta = \delta\rho_m/\rho_m$ with ρ_m and $\delta\rho_m$ being the density and the overdensity of matter respectively; the prime denotes the derivative with respect to the conformal time τ, and f_ν reads as

$$f_\nu \equiv \frac{\Omega_\nu h^2}{\Omega_m h^2} = \frac{1}{\Omega_m h^2}\frac{\sum_i m_{\nu,i}}{94.14\text{eV}}. \tag{33}$$

In an Einstein-de Sitter universe, one has a simple solution of the Eq. (32)

$$\delta \propto a^p, \tag{34}$$

where $p \approx 1 - 3f_\nu/5$.

From $\delta \propto a^p$, one can derive the suppression of $P(k)$ goes as follows: the growth from matter-radiation equality epoch a_{eq} to the present a_0 is

$$\frac{\delta(a_0)}{\delta(a_{eq})} = (1+z_{eq})(1+z_{eq})^{-\frac{3}{5}f_\nu} = (1+z_{eq})e^{-\frac{3}{5}f_\nu \ln(1+z_{eq})}. \tag{35}$$

The power spectrum $P(k)$ is the variance of the fluctutions δ in Fourier space, so massive neutrinos suppress it by the same factor as it suppresses δ^2, i.e.

$$\frac{P(k,f_\nu) - P(k,f_\nu=0)}{P(k,f_\nu=0)} \simeq -\frac{6}{5}f_\nu \ln(1+z_{\rm eq}). \tag{36}$$

We assume that Ω_m is the same for the scenarios with and without massive neutrinos, hence $(1+z_{eq}) \approx 23900\Omega_m h^2$. For the *Planck* 2015 results, $\Omega_m h^2 = 0.14205$. This gives the suppression of the matter power spectrum $\frac{\Delta P}{P} \approx -8.4 f_\nu$.

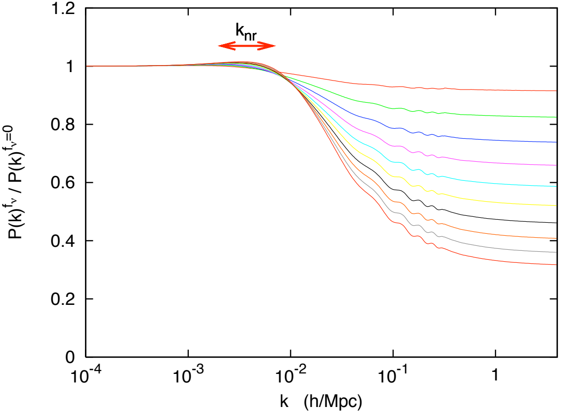


Figure: Ratio of the matter power spectrum including three degenerate massive neutrinos with density fraction f_ν to that with three massless neutrinos. The parameters $(\omega_{\rm m}, \Omega_\Lambda) = (0.147, 0.70)$ are kept fixed, and from top to bottom the curves correspond to $f_\nu = 0.01, 0.02, 0.03, \ldots, 0.10$. The individual masses m_ν range from 0.046 to 0.46 eV, and the scale $k_{\rm nr}$ from $2.1 \times 10^{-3} h\,\text{Mpc}^{-1}$ to $6.7 \times 10^{-3} h\,\text{Mpc}^{-1}$ as shown on the top of the figure. *J. Lesgourgues, S. Pastor, 2006,PR,* ***429*** *307*

Neutrinos Mass Hierarchy Parameter $\Delta = (m_3 - m_1)/(m_1 + m_3)$

Since $|\Delta m_{31}^2| \equiv |m_3^2 - m_1^2| = 2.5 \times 10^{-3}$ eV^2, we introduce a new parameter, named the neutrino mass hierarchy parameter,

$$\Delta = \frac{m_3 - m_1}{m_1 + m_3} \tag{37}$$

to measure the neutrino mass hierarchy. The positive (negative) sign of Δ denotes the normal (inverted) mass hierarchy. This mass hierarchy parameter is dimensionless and can vary in the range $[-1, 1]$. Taking into account the neutrino oscillation experiments, the three neutrino mass eigenvalues are given by

$$m_1 = \frac{1-\Delta}{2\sqrt{|\Delta|}}\sqrt{|\Delta m_{31}^2|}, \tag{38}$$

$$m_2 = \sqrt{m_1^2 + \Delta m_{21}^2}, \tag{39}$$

$$m_3 = \frac{1+\Delta}{2\sqrt{|\Delta|}}\sqrt{|\Delta m_{31}^2|}. \tag{40}$$

The minimum eigenvalue of neutrino masses is m_1 for $\Delta > 0$ (NH) or m_3 for $\Delta < 0$ (IH). The lightest neutrino is massless if $\Delta = 1$ for NH or $\Delta = -1$ for IH. From the above equations, the total neutrino mass is given by

$$\sum_\nu m_\nu = \sqrt{\frac{|\Delta m_{31}^2|}{|\Delta|}} + \sqrt{\frac{(1-\Delta)^2}{4|\Delta|}|\Delta m_{31}^2| + \Delta m_{21}^2} \quad \Omega_\nu = \frac{\sum_\nu m_\nu}{93.14h^2\ \mathrm{eV}}. \tag{41}$$

The total neutrino mass as a function of Δ is illustrated as

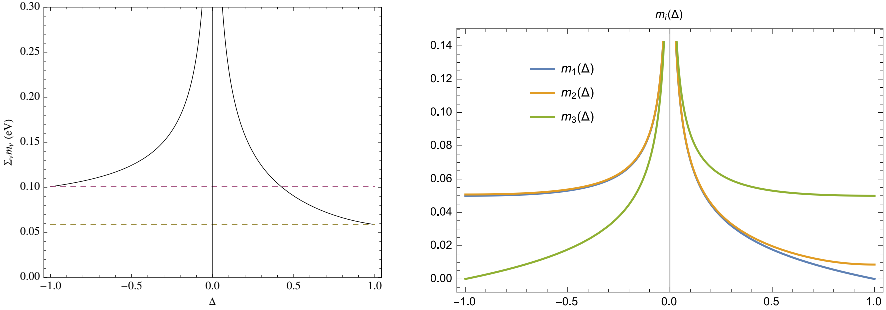
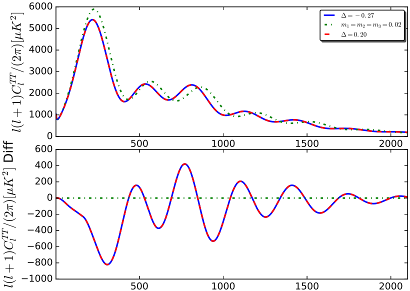
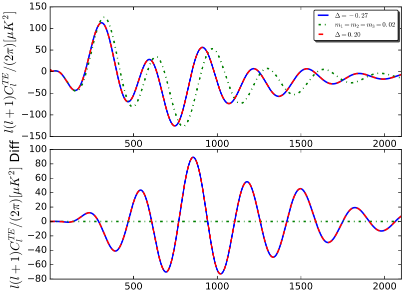
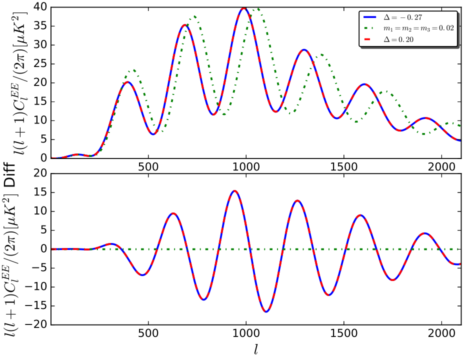


Figure: The total neutrino mass vs hierarchy parameter Δ.


$\Delta = -0.27$
$m_1 = m_2 = m_3 = 0.02$
$\Delta = 0.20$
$l(l+1)C_l^{TT}/(2\pi)[\mu K^2]$
$l(l+1)C_l^{TT}/(2\pi)[\mu K^2]$ Diff


$\Delta = -0.27$
$m_1 = m_2 = m_3 = 0.02$
$\Delta = 0.20$
$l(l+1)C_l^{TE}/(2\pi)[\mu K^2]$
$l(l+1)C_l^{TE}/(2\pi)[\mu K^2]$ Diff


$\Delta = -0.27$
$m_1 = m_2 = m_3 = 0.02$
$\Delta = 0.20$
$l(l+1)C_l^{EE}/(2\pi)[\mu K^2]$
$l(l+1)C_l^{EE}/(2\pi)[\mu K^2]$ Diff
l

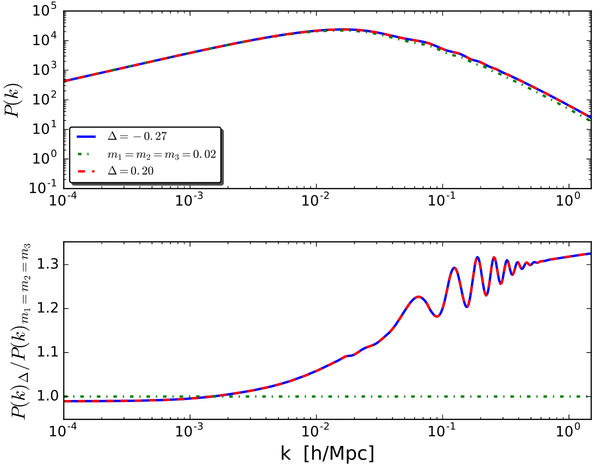


Figure: The effects of hierarchy parameter Δ on the matter power spectrum at $z = 0$.

Using *Planck* 2015 LowTEB, TT, TE, EE + BAO DR12(6DFGS, MGS BOSS DR12 CMASS and LOWZ and RSD) + JLA SN + HST 2016($H_0 = 73.03 \pm 1.79 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$) for the six-parameter based ΛCDM model.

Parameters	Flat Priors	68% limits
$\Omega_b h^2$	$[0.005, 0.1]$	0.02237 ± 0.00014
$\Omega_c h^2$	$[0.001, 0.99]$	0.1177 ± 0.0010
$100\theta_{MC}$	$[0.5, 10]$	1.04095 ± 0.00031
τ	$[0.01, 0.8]$	0.078 ± 0.016
$\ln(10^{10} A_s)$	$[2, 4]$	3.086 ± 0.032
n_s	$[0.8, 1.2]$	0.9697 ± 0.0038
H_0	$[40, 100]$	67.93 ± 0.48
Ω_Λ	–	0.6942 ± 0.0062
Ω_m	–	0.3058 ± 0.0062
σ_8	–	0.815 ± 0.013
z_{re}	–	$9.89^{+1.57}_{-1.36}$
Δ (95%)	$[-1, 1]$	$-1 \leq \Delta < -0.40$ or $0.32 < \Delta \leq 1$
$m_{\nu,\mathrm{min}}^{\mathrm{NH}}$ eV (95%)	–	< 0.030
$m_{\nu,\mathrm{min}}^{\mathrm{IH}}$ eV (95%)	–	< 0.024

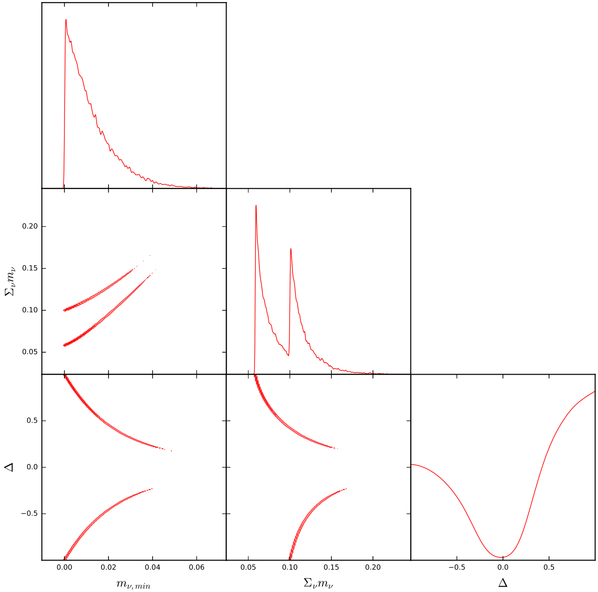


Figure: The 1D marginalized distribution and 2D contours for interested model parameters with 68% and 95% confidence levels.

Summary

1. We propose a parameter ($\Delta = (m_3 - m_1)/(m_1 + m_3)$) to measure the neutrinos mass hierarchy, namely the positive (negative) sign of Δ for normal (inverted) mass hierarchy. All of the neutrino masses are determined by Δ if the results of neutrino oscillation experiments are utilized

2. We find that both the CMB and matter power spectra are sensitive to the total neutrino mass, not the mass hierarchy.

3. The normal hierarchy is slightly preferred due to the fact that the lower total neutrino mass provides a slightly better fit to the current cosmological data.

Thank You for Your Attention!

In 1950s, Bruno Pontecorvo first proposed neutrino oscillations inspired by the $K^0 - \overline{K^0}$ system. Denoting $\mathcal{U}$ the $(3+N_s)\times(3+N_s)$ square mixing matrix, a particular flavour state can be written as

$$|\nu_\alpha\rangle = \sum_{k=1}^{3+N_s} \mathcal{U}^*_{\alpha k}|\nu_k\rangle, \quad \alpha = e, \mu, \tau, s_1, ..., s_{N_s}, \tag{42}$$

with the normalization conditions $\langle\nu_\alpha|\nu_\beta\rangle = \delta_{\alpha\beta}$ and $\langle\nu_i|\nu_j\rangle = \delta_{ij}$. Massive neutrinos are Hamiltonian eigenstates, $|\nu_\alpha\rangle$ evolves in time as

$$|\nu_\alpha(t)\rangle = \sum_{k=1}^{3+N_s} \mathcal{U}^*_{\alpha k} e^{-iE_k t}|\nu_k\rangle. \tag{43}$$

For unitary $\mathcal{U}^\dagger\mathcal{U} = 1$, one also has $|\nu_k\rangle = \sum_\beta \mathcal{U}_{\beta k}|\nu_\beta\rangle$, and one has

$$|\nu_\alpha(t)\rangle = \sum_\beta \left(\sum_{k=1}^{3+N_s} \mathcal{U}^*_{\alpha k} e^{-iE_k t}\mathcal{U}_{\beta k}\right)|\nu_\beta\rangle, \tag{44}$$

thus after restriction to active states $\mathcal{U} \to U$, the coefficient proportional to $|\nu_\beta\rangle$ is

$$\langle\nu_\beta|\nu_\alpha(t)\rangle = \sum_{k=1}^{3+N_s} U^*_{\alpha k} e^{-iE_k t} U_{\beta k}. \tag{45}$$

Thus one gets the probability that on performing a measurement of neutrino flavour at time t one detects a flavour β

$$P_{\nu_\alpha \to \nu_\beta} = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{k,j=1}^{3+N_s} U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j} e^{-i(E_k - E_j)t}. \tag{46}$$

For relativistic neutrinos

$$E_i = \sqrt{|\vec{p}|^2 + m_i^2} \approx |\vec{p}| + \frac{m_i^2}{2|\vec{p}|}, \tag{47}$$

one gets

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{k,j=1}^{3+N_s} U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj}}{2|\vec{p}|} t\right). \tag{48}$$

where $\Delta m^2_{kj} \equiv m_k^2 - m_j^2$. Replacing the time t with distance L/c, one obtains

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_{k,j=1}^{3+N_s} U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj}}{2|\vec{p}|} L\right). \tag{49}$$

Since $\sum_{k,j=1}^{3+N_s} = \sum_{k>j}^{3+N_s} + \sum_{k=j}^{3+N_s} + \sum_{k<j}^{3+N_s}$, one has

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_{k=1}^{3+N_s} |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re}\left[\sum_{k>j}^{3+N_s} U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj}}{2|\vec{p}|} L\right)\right]. \tag{50}$$

Using the fact $P_{\nu_\alpha \to \nu_\beta}(L=0) = \sum_{k,j=1}^{3+N_s} U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j} = \delta_{\alpha\beta}$ and $\cos(2\theta) = 1 - 2\sin^2(\theta)$, the above equation can also be rewritten as

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta}(L) &= \delta_{\alpha\beta} - 4\sum_{k>j} \mathrm{Re}\left[U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \sin^2\left(\frac{\Delta m^2_{kj}}{4|\vec{p}|} L\right) \\ &+ 2\sum_{k>j} \mathrm{Im}\left[U^*_{\alpha k} U_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \sin\left(\frac{\Delta m^2_{kj}}{2|\vec{p}|} L\right). \end{aligned} \tag{51}$$

The *survival probability* for $\alpha = \beta$ is

$$P_{\nu_\alpha \to \nu_\alpha}(L) = 1 - 4\sum_{k>j} |U_{\alpha k}|^2 |U_{\alpha j}|^2 \sin^2\left(\frac{\Delta m^2_{kj}}{2|\vec{p}|} L\right). \tag{52}$$

And the *total transition probability* is $1 - P_{\nu_\alpha \to \nu_\alpha}(L)$.

If the neutrino production or detection occurs on a distance whose uncertainty is much larger than the oscillation lengths $L_{kj}^{osc} = 4\pi|\vec{p}|/\Delta m_{kj}^2$, the expression should be averaged over distances larger than L_{kj}^{osc}, one has

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_{k=1}^{3+N_s} |U_{\alpha k}|^2 |U_{\beta k}|^2. \tag{53}$$

Similarly, for the antineutrinos,

$$P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta}(L) = \sum_{k,j=1}^{3+N_s} U_{\alpha k}^* U_{\beta k} U_{\alpha j}^* U_{\beta j} \exp\left(-i\frac{\Delta m_{kj}^2}{2|\vec{p}|}L\right), \tag{54}$$

and the difference

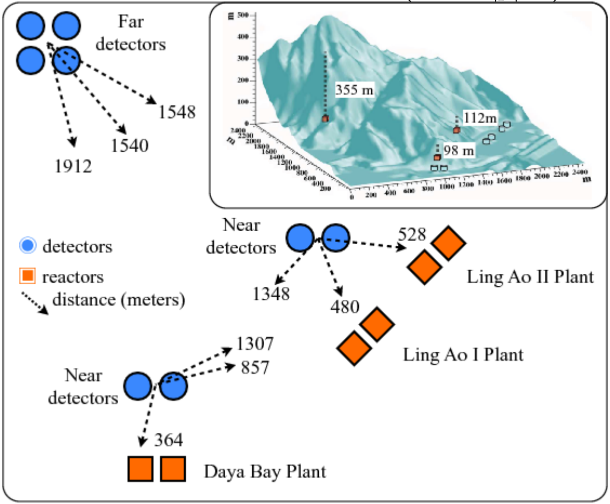
$$P_{\nu_\alpha \to \nu_\beta}(L) - P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta}(L) = 4\sum_{k>j} \mathrm{Im}\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j}^* U_{\beta j}\right] \sin\left(\frac{\Delta m_{kj}^2}{2|\vec{p}|}L\right) \tag{55}$$

can be used to check the CP violation ($|\nu_\alpha\rangle \xleftrightarrow{\mathrm{CP}} |\overline{\nu}_\alpha\rangle$). That can only be found in flavour transition processes due to the fact $\mathrm{Im}\left[U_{\alpha k}^* U_{\alpha k} U_{\alpha j}^* U_{\alpha j}\right] = 0$ for any k, j.

Daya Bay Experiments for Illustration

Using $|\Delta m^2_{32}| \approx |\Delta m^2_{31}| \gg \Delta m^2_{21}$ and $|U_{e3}|^2 = \sin^2\theta_{13} \ll 1$, one has

$$P_{\overline{\nu}_e \to \overline{\nu}_e}(L) = 1 - \sin^2 2\theta_{13} \sin^2\left(1.27 \frac{\Delta m^2_{31} L}{|\vec{p}|}\right) \tag{56}$$



In the global fit for the whole set of parameters

$$\left(\delta m^2, \sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, |\Delta m^2|, \delta/\pi\right) \tag{57}$$

Results of the global 3ν oscillation analysis. *Fogli,G.L.,etal.2012.Phys.Rev.D,86,013012.*

Parameter	Best fit	1σ range	2σ range	3σ range
$\delta m^2/10^{-5}$ eV^2 (NH or IH)	7.54	7.32–7.80	7.15–8.00	6.99–8.18
$\sin^2\theta_{12}/10^{-1}$ (NH or IH)	3.07	2.91–3.25	2.75–3.42	2.59–3.59
$\Delta m^2/10^{-3}$eV^2 (NH)	2.43	2.33–2.49	2.27–2.55	2.19–2.62
$\Delta m^2/10^{-3}$eV^2 (IH)	2.42	2.31–2.49	2.26–2.53	2.17–2.61
$\sin^2\theta_{13}/10^{-2}$ (NH)	2.41	2.16–2.66	1.93–2.90	1.69–3.13
$\sin^2\theta_{13}/10^{-2}$ (IH)	2.44	2.19–2.67	1.94–2.91	1.71–3.15
$\sin^2\theta_{23}/10^{-1}$ (NH)	3.86	3.65–4.10	3.48–4.48	3.31–6.37
$\sin^2\theta_{23}/10^{-1}$ (IH)	3.92	3.70–4.31	3.53–4.84 $\oplus$ 5.43–6.41	3.35–6.63
δ/π (NH)	1.08	0.77–1.36	–	–
δ/π (IH)	1.09	0.83–1.47	–	–

where $\delta m^2 = m_2^2 - m_1^2$ and $\Delta m^2 = m_3^2 - \frac{m_2^2+m_1^2}{2}$. And, $\Delta m^2 > 0 (< 0)$ corresponds to the normal (inverted) mass spectrum hierarchy.